3. BALABUKH L.I. and USYUKIN V.I., Approximate theory of soft shells of revolution. Proceeding of the VIII-th National Conference on the Theory of Shells and Plates. Rostov-on-Don, 1971, Moscow, 1973.
4. GRIGOR'YEV A.S., Equilibrium state of a zero-moment shell of revolution under large deformations. PMM 25, 6, 1961.
5. USYUKIN V.I., On the equations of the theory of large deformations of soft shells. Izv. Akad. Nauk SSSR, MTT, 1, 1976.
6. MAGULA V.E., Elastic Ship Structures. Leningrad, Sudostroenie, 1978.
7. MAGULA V.E., Basic relationships of the theory of soft shells. Tr. Nikolaev. korablestroit in-ta, 78, 1973.
8. CHERNYKH K.F., Non-linear theory of isotropically elastic thin shells. Izv. Akad. Nauk SSSR, MTT, 2, 1980.
9. CHERNYKH K.F., Non-linear theory of thin elastic shells made of elastomers. In books: Statbility of Plates and Shells. Saratov, Izd-vo Sarat. un-ta, 1981.
10. NOVOZHILOV V.V., Theory of Thin Shells. Leningrad, Sudpromgiz, 1951.
11. CHERNYKH K.F., Linear Theory of Shells. 1, Leningrad, Izt-vo LGU, 1962.

Translated by L.K.

PMM U.S.S.R.,Vol.49,No.6,pp. 763-770,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

ON THERMOELASTIC STRESSES IN AN ASYMMETRICALLY HEATED HALF-SPACE*

L.N. GERMANOVICH, L.V. ERSHOV and I.D. KILL

A quasistatic problem of thermoelasticity is considered for a half-space in the case of convective heat exchange (boundary condition of the third kind). In the case of boundary conditions of the first and second kind all results are obtained in exactly the same manner. The exact solution of the problem is found in the form allowing the construction of an approximate solution, simple and suitable for numerical computations and based on the asymptotic expansion of the temperature and the stresses as $t \to 0$. The problem is reduced to determining single integrals of simple functions, and in many cases the integrals can be expressed in terms of elementary functions. The error of the approximate solution is estimated.

Unlike the results obtained earlier in /1—3/, the temperature distribution in the medium adjacent to the half-space is not assumed to be axisymmetric, i.e. a general asymmetric distribution is studied under certain constraints that are not significant from the physical point of view. Such asymmetric distributions are very common in practice /4/. The results of this paper can be used to study the fracture of brittle materials which can occur under the action of thermoelastic stresses /5/.

It should be noted that application of the numerical methods which were successfully used in solving the symmetric problem of thermoelasticity /6/ encounters, in the case of asymmetric, obvious difficulties caused by the increased dimensionality of the problem.

1. The initial temperature of the elastic half-space $z \geqslant 0$ and the medium filling the region $z < 0$ is $T = 0$. At the instant $t = 0$ the temperature of the medium rises instantaneously and assumes the distribution (r, φ, z are cylindrical coordinates)

$$\Theta = \Theta(r, \varphi), \quad \Theta(r, \varphi + 2\pi) = \Theta(r, \varphi) \tag{1.1}$$

and the function $\Theta(r, \varphi)$ can be written in the form of a Fourier series whose coefficients admit of the n-th order Hankel transformation in r

$$\Theta(r, \varphi) = \sum_{n=0}^{\infty} [\vartheta_n(r) \cos n\varphi + \tau_n(r) \sin n\varphi], \quad \tau_0(r) \equiv 0 \tag{1.2}$$

$$\vartheta_n{}^H(\lambda) = H_\lambda[\vartheta_n(r)] = \int_0^\infty r\vartheta_n(r) J_n(\lambda r)\, d\lambda$$

$$\tau_n{}^H(\lambda) = H_\lambda[\tau_n(r)], \quad n = 0, 1, 2,$$

*Prikl.Matem.Mekhan.,49,6,1001-1009,1985 where J_n is the n-th order Bessel function of the first kind. In the physical problems the cnditions for the existence of representations (1.2) hold as a rule.

We require to find the temperature and stress fields inside the half-space when the heat exchange with the medium occupying the region $z < 0$ obeys Newton's law.

2. Let us transfer to dimensionless coordinates, putting $r' = r/\delta$, $z' = z/\delta$, $t' = at/\delta^2$, $h' = h\delta$ where a is the thermal diffusivity, h is the relative heat transfer constant and δ is a characteristic dimension. Neglecting, for simplicity, the primes accompanying the dimensionless quantities, we shall write the boundary value problem of heat conduction as follows:

$$\frac{\partial T}{\partial t} = \Delta T \quad \left(\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right) \tag{2.1}$$

$$T|_{t=0} = 0; \quad \frac{\partial T}{\partial z}\Big|_{z=0} = h(T|_{z=0} - \Theta); \quad T(\infty, \varphi, z, t) = T(r, \varphi, \infty, t) = 0$$

Applying to the problem a Laplace transform in t, we obtain

$$sT^* = \Delta T^*; \quad \frac{\partial T^*}{\partial z}\Big|_{z=0} = h\left(T^*|_{z=0} - \frac{\Theta}{s}\right) \tag{2.2}$$

$$T^*(r, \varphi, z, s) = L_t[T(r, \varphi, z, t)] = \int_0^\infty T(r, \varphi, z, t)e^{-st}\,dt$$

We shall seek the solution of problem (2.2) in the form

$$T^*(r, \varphi, z, s) = \sum_{n=0}^{\infty} [u_n^*(r, z, s)\cos n\varphi + v_n^*(r, z, s)\sin n\varphi], \quad v_0^* \equiv 0 \tag{2.3}$$

Substituting (1.2) and (2.3) into the equation and boundary condition (2.2) and equating the coefficients of like harmonics, we obtain

$$\frac{\partial^2 u_n^*}{\partial r^2} + \frac{1}{r}\frac{\partial u_n^*}{\partial r} + \frac{\partial^2 u_n^*}{\partial z^2} - \frac{n^2}{r^2}u_n^* = su_n^* \tag{2.4}$$

$$\frac{\partial u_n^*}{\partial z}\Big|_{z=0} = h\left(u_n^*|_{z=0} - \frac{\vartheta_n}{s}\right), \quad n = 0, 1, 2, \ldots$$

and anlogous boundary value problems for determining v_n^*, $n = 1, 2, \ldots$

Let us now apply the Hankel transform of the n-th order in r. This yields a boundary value problem. Solving it and inverting the Hankel transform, we obtain

$$u_n^*(r, z, s) = h\int_0^\infty \lambda J_n(\lambda r)s\xi\vartheta_n^H(\lambda)\,d\lambda \tag{2.5}$$

$$\xi = \frac{\exp(-z\sqrt{s+\lambda^2})}{s^2(\sqrt{s+\lambda^2}+h)}, \quad n = 0, 1, 2, \ldots$$

Finding in the same manner $v_n^*(r, z, s)$ and subsituting it into (2.3), we obtain

$$T^*(r, z, s) = \sum_{n=0}^{\infty} T_n^*(r, z, s), \quad T_n^* = h\int_0^\infty \lambda J_n(\lambda r)s\xi\omega_{1n}\,d\lambda \tag{2.6}$$

$$\omega_{1n} = \vartheta_n^H(\lambda)\cos n\varphi + \tau_n^H(\lambda)\sin n\varphi, \; n = 0, 1, 2, \ldots$$

3. Let us find an expression for the thermoelastic displacement potential. Following /7/, we have (ν is Poisson's ratio, α is the coefficient of linear expansion)

$$\Phi^*(r, z, s) = \frac{1+\nu}{1-\nu}\frac{\alpha}{s^2}(sT^* - \lim_{s\to 0} sT^*) = \tag{3.1}$$

$$\alpha h\frac{1+\nu}{1-\nu}\sum_{n=0}^{\infty}\int_0^\infty \lambda J_n(\lambda r)\left[\xi - \frac{e^{-\lambda z}}{s^2(\lambda+h)}\right]\omega_{1n}\,d\lambda$$

We find the expressions for the stresses corresponding to (3.1) using the formulas for the stresses from /8/, having previously transformed them to cylindrical coordinates (μ is the shear modulus)

$$p_{rr} = 2\mu\left(\frac{\partial^2\Phi}{\partial r^2} - \Delta\Phi\right), \quad p_{\varphi\varphi} = 2\mu\left(\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} - \Delta\Phi\right) \tag{3.2}$$

$$p_{zz} = 2\mu\left(\frac{\partial^2\Phi}{\partial z^2} - \Delta\Phi\right), \quad p_{r\varphi} = 2\mu\left(\frac{1}{r}\frac{\partial^2\Phi}{\partial r\,\partial\varphi} - \frac{1}{r^2}\frac{\partial\Phi}{\partial\varphi}\right)$$

$$p_{rz} = 2\mu\frac{\partial^2\Phi}{\partial r\,\partial z}, \quad p_{\varphi z} = 2\mu\frac{1}{r}\frac{\partial^2\Phi}{\partial\varphi\,\partial z}$$

As a result owe obtain expressions for the stresses p_{ij}^* ($i, j = r, \varphi, z$). We also find that the expressions for p_{zz}^*, p_{rz}^* and $p_{\varphi z}^*$ are different from zero on the free surface $z = 0$. We remove them by bringing in an additional "temperature-free" solution obtained with help of the Galerkin function G. If the form of its representation is known, then the constants occurring in it can be found from the equations

$$q^*_{zz}\big|_{z=0} = -p^*_{zz}\big|_{z=0}, \quad q^*_{rz}\big|_{z=0} = -p^*_{rz}\big|_{z=0}, \quad q^*_{\varphi z}\big|_{z=0} = -p^*_{\varphi z}\big|_{z=0} \tag{3.3}$$

where q_{ij}^*, $(i, j = r, \varphi, z)$ are the stresses corresponding to the Galerkin function, which can be found using the formulas in /8/.

After substituting the values of p_{ij}^* and q_{ij}^* into (3.3), we reduce the last two equations to the form $\partial F^*/\partial r = 0$; $\partial F^*/\partial\varphi = 0$; system (3.3) then reduces to the following two equations:

$$\left\{\frac{\partial}{\partial z}\left[(2-\nu)\Delta G^* - \frac{\partial^2 G^*}{\partial z^2}\right] + (1-2\nu)\left(\frac{\partial^2\Phi^*}{\partial z^2} - \Delta\Phi^*\right)\right\}\Big|_{z=0} = 0 \tag{3.4}$$

$$\left\{(1-\nu)\Delta G^* - \frac{\partial^2 G^*}{\partial z^2} + (1-2\nu)\frac{\partial\Phi^*}{\partial z}\right\}\Big|_{z=0} = 0$$

We shall seek the representation of the Galerkin function in the form

$$G^*(r,\varphi,z,s) = \sum_{n=0}^{\infty}\Big\{\cos n\varphi\int_0^\infty [A_n(s,\lambda) + zB_n(s,\lambda)]\,J_n(\lambda r)\,e^{-\lambda z}\,d\lambda + \tag{3.5}$$

$$\sin n\varphi\int_0^\infty [C_n(s,\lambda) + zD_n(s,\lambda)]\,J_n(\lambda r)\,e^{-\lambda z}\,d\lambda\Big\}$$

We can confirm directly that G^* is a biharmonic function. Subsituting (3.1) and (3.5) into (3.4) and equating the coefficients of like harmonics, we obtain a system of linear equations for computing A_n, B_n, C_n, D_n $(n = 0, 1, 2, \ldots)$. Thus the function G^* will be completely determined. Calculating the expressions for q_{ij}^* and combining them with the corresponding p_{ij}^*, we obtain the expressions for the total stresses σ_{ij}^*

$$\sigma_{ij}^* = \sum_{n=0}^{\infty}\sigma_{ij}^{n*}, \quad i, j = r, \varphi, z \tag{3.6}$$

$$\frac{\sigma_{rr}^{n*}}{D} = -T_n^* + h\int_0^\infty\Big\{\frac{e^{-\lambda z}}{s^2}\Big[\frac{J_{n+1}(\lambda r)}{r}\lambda k_1 +$$

$$J_n(\lambda r)\frac{n^2-n}{r^2}k_1 + J_n(\lambda r)(2-\lambda z)\lambda^2\Big] + \frac{J_{n+1}(\lambda r)}{r}\lambda k_2 +$$

$$J_n(\lambda r)\frac{n^2-n}{r^2}k_2 - J_n(\lambda r)[\lambda^3\xi -$$

$$(2h - \lambda + h\lambda z + \lambda^2 z)\lambda^2\eta]\Big\}\,\omega_{1n}\,d\lambda$$

$$\frac{\sigma_{\varphi\varphi}^{n*}}{D} = -T_n^* - h\int_0^\infty\Big\{\frac{e^{-\lambda z}}{s^2}\Big[\frac{J_{n+1}(\lambda r)}{r}\lambda k_1 + J_n(\lambda r)\frac{n^2-n}{r^2}k_1 -$$

$$2J_n(\lambda r)\nu\lambda^2\Big] + \frac{J_{n+1}(\lambda r)}{r}\lambda k_2 + J_n(\lambda r)\frac{n^2-n}{r^2}k_2 +$$

$$2J_n(\lambda r)(h+\lambda)\nu\lambda^2\eta\Big\}\,\omega_{1n}\,d\lambda$$

$$\frac{\sigma_{zz}^{n*}}{D} = h\int_0^\infty J_n(\lambda r)\Big[\frac{ze^{-\lambda z}}{s^2} + \xi - (1 + hz + \lambda z)\,\eta\Big]\lambda^3\omega_{1n}\,d\lambda$$

$$\frac{\sigma_{r\varphi}^{n*}}{D} = h\frac{n}{r^2}\int_0^\infty [(n-1)J_n(\lambda r) - \lambda rJ_{n+1}(\lambda r)]\Big(\frac{k_1}{s^2}e^{-\lambda z} + k_2\Big)\omega_{2n}\,d\lambda$$

$$\frac{\sigma_{rz}^{n*}}{D} = h\int_0^\infty\Big[\frac{n}{\lambda r}J_n(\lambda r) - J_{n+1}(\lambda r)\Big]k_3\lambda^2\omega_{1n}\,d\lambda$$

$$\frac{\sigma_{\varphi z}^{n*}}{D} = h\frac{n}{r}\int_0^\infty J_n(\lambda r)\,k_3\lambda\,\omega_{2n}\,d\lambda$$

$$k_1 = 2\nu - 2 + \lambda z, \quad k_2 = \lambda\xi - [2h(\nu-1) + (2\nu - 1 + hz)\lambda - \lambda^2 z]\eta$$

$$k_3 = \frac{1-\lambda z}{s^2}e^{-\lambda z} + h\xi - \frac{e^{-\omega z}}{s^2} - (h - h\lambda z - \lambda^2 z)\,\eta$$

$$\eta = \frac{e^{-\lambda z}}{s^2(\omega+h)}, \quad \omega = \sqrt{s+\lambda^2}, \quad \omega_{2n} = \tau_n^H(\lambda)\cos n\varphi - \vartheta_n^H(\lambda)\sin n\varphi$$

$$D = \frac{2\mu\alpha(1+\nu)}{1-\nu}$$

Using a table from /1/, we can write the originals for the temperature and the stresses. However, since the solution was constructed in a formal manner, the convergence of the integrals obtained must be checked. This requirement is practically trouble-free, because of the presence of the factors $\exp(-\lambda z)$ and $\exp(-\lambda^2 t)$ in the integrands. (Such a check was carried out for the example for the boundary distribution (1.1) given below). The singularities in the integrands can be eliminated.

4. The exact solution obtained above contains removable singularities which hamper its application and can cause considerable errors in numerical computations. In this connection we shall investigate the asymptotic behaviour of the solution as $t \to 0$. To simplify the problems connected with the convergence of series, we shall confine ourselves to the case when the boundary temperature distribution (1.1) is represented by a finite trigonometric polynomial, i.e. the series in (1.2) is replaced by a finite sum up to and including N. This restriction is not significant from the physical point of view.

Let us first investigate the temperature distribution, considering separately every harmonic T_n^* in (2.6). Inverting T_n^*, we obtain the relation

$$T_n = \int_0^\infty \lambda J_n(\lambda r)\,\omega_{1n}\,d\lambda \int_0^t \exp(-\lambda^2\tau)\,L_\tau^{-1}\left[\frac{h\exp(-z\sqrt{s})}{\sqrt{s}+h}\right]d\tau \tag{4.1}$$

where L_τ^{-1} is an operator inverse to L_s (the lower index denotes the argument of the original). Using Taylor's formula for $\exp(-\lambda^2\tau)$, we obtain

$$T_n = \int_0^\infty \lambda J_n(\lambda r)\,\omega_{1n}\,d\lambda \int_0^t f_0(z,\tau)\sum_{m=0}^{M}\frac{(-1)^m\lambda^{2m}}{m!}\tau^m\,d\tau + T_n^M \tag{4.2}$$

$$T_n^M = \frac{(-1)^{M+1}}{(M+1)!}\int_0^\infty \lambda^{2M+3}J_n(\lambda r)\,\omega_{1n}\exp(-\lambda^2\zeta)\,d\lambda\int_0^t \tau^{M+1}f_0(z,\tau)\,d\tau$$

$$f_m(z,t) = L_t^{-1}\left[\frac{h\exp(-z\sqrt{s})}{s^m(\sqrt{s}+h)}\right],\quad m = 0, 1, 2, \ldots$$

$$0 < \zeta < \tau,\ M = 0, 1, 2, \ldots$$

In determining the asymptotic expansion we choose the same system of functions both for the approximation and the comparison /9, 10/

$$\mu_m(t) = (-1)^m\int_0^t f_0(z,t)\,\tau^m\,d\tau,\quad m = 0, 1, 2, \ldots \tag{4.3}$$

From (4.2) and (4.3) we see that $\mu_{m+1}(t) = o(\mu_m(t))$ and $T_n^m = O(\mu_{m+1}(t))$ as $t \to 0$, $m = 0, 1, 2, \ldots$. Therefore we have the following asymptotic expansion:

$$T_n = \sum_{m=0}^\infty \frac{A_{1,n}^{2m+1,n}\mu_m(t)}{m!},\quad A_{j,k}^{i,n}(r,\varphi) = \int_0^\infty \lambda^i J_k(\lambda r)\,\omega_{jn}\,d\lambda \tag{4.4}$$

Using mathematical induction and the formulas from /11/, we obtain the following relations for $\mu_m(t)$ and $f_m(z,t)$

$$\mu_m(t) = (-1)^m m!\sum_{k=0}^m \frac{(-1)^k t^{m-k}}{(m-k)!}f_{k-1}(z,t) \tag{4.5}$$

$$f_m(z,t) = \frac{f_{m-1}(z,t)}{h^2} - \frac{\varphi_{2m-1}(z,t)}{h} + \varphi_{2m}(z,t),\quad m = 1, 2, \ldots$$

$$f_0(z,t) = \frac{h}{\sqrt{\pi t}}\exp\left(-\frac{z^2}{4t}\right) - h^2\exp(h^2t + hz)\,\mathrm{erfc}\,\frac{z+2ht}{2\sqrt{t}}$$

$$\varphi_m(z,t) = L_t^{-1}\left[\frac{\exp(-z\sqrt{s})}{s^{m/2}}\right] = (4t)^{(m-1)/2}\,i^{m-1}\,\mathrm{erfc}\,\frac{z}{2\sqrt{t}}$$

$$m = 2, 3, \ldots$$

$$\varphi_1(z,t) = \frac{1}{\sqrt{\pi t}}\exp\left(-\frac{z^2}{4t}\right) \tag{4.6}$$

$$i^m\,\mathrm{erfc}\,x = -\frac{x}{m}\,i^{m-1}\,\mathrm{erfc}\,x + \frac{1}{2m}\,i^{m-2}\,\mathrm{erfc}\,x,\quad m = 1, 2, \ldots$$

$$i^{-1}\,\mathrm{erfc}\,x = \frac{2}{\sqrt{\pi}}\exp(-x^2),\quad i^0\,\mathrm{erfc}\,x = \mathrm{erfc}\,x$$

Thus we obtain the approximate solution (which becomes asymptotically exact as $t \to 0$)

$$T(r,\varphi,z,t) = \sum_{n=0}^N\sum_{m=0}^M\sum_{k=0}^m \frac{(-1)^{m+k}t^{m-k}}{(m-k)!}A_{1,n}^{2m+1,n}(r,\varphi)\,f_{k-1}(z,t) \tag{4.7}$$

more suited for carrying out the calculations for small t than the exact solution. The functions $f_{k-1}(z,t)$ contain only a single, special, well tabulated function, namely the auxilliary probability integral. The integrals $A_{1,n}^{2m+1,n}$ converge rapidly and are often taken in terms of elementary functions (see the example).

Let us now analyse the stress. From formulas (3.6) we see that the construction of the asymptotic forms of the stresses σ_{rr}, $\sigma_{\varphi\varphi}$, σ_{zz}, $\sigma_{r\varphi}$ reduces to constructing the asymptotic expansions, as $t \to 0$, of the expressions

$$I = \int_0^\infty \lambda^i J_j(\lambda r)\, \omega_{pn} \exp(-q\lambda z)\, d\lambda \int_0^t d\vartheta \int_0^\vartheta \exp(-\lambda^2\tau) f_0(z, \tau)\, d\tau \tag{4.8}$$
$$(i = 1, 2, 3, 4;\ j = 0, 1, 2 \ldots;\ p = 1, 2;\ q = 0, 1)$$

In the case of the stresses σ_{rz} and $\sigma_{\varphi z}$ we must also construct the asympotic expansion as $t \to 0$ of the expression of the form (4.8) where $f_0(z, \tau)$ must be replaced by $L_\tau^{-1}[e^{-z\sqrt{s}}]$ and we assume that $i = 1, 2$; $j = 0, 1, 2, \ldots$; $p = 2$; $q = 0$. Since these cases are completely analogous, we shall only consider (4.8). We will merely note that the operation $L_\tau^{-1}[e^{-z\sqrt{s}}]$ and $z = 0$ is defined only in the class of generalized functions. This causes a certain amount of inconvenience, but since we know the exact values of σ_{rz} and $\sigma_{\varphi z}$ when $z = 0$, namely $(\sigma_{rz} = \sigma_{\varphi z} = 0)$, we shall construct their asymptotic expansions under the assumption that $z > 0$.

Chossing the following system of functions for approximation and comparison:

$$\mu_m(t) = (-1)^m \int_0^t d\vartheta \int_0^\vartheta \tau^m f_0(z, \tau)\, d\tau \tag{4.9}$$

and expanding $\exp(-\lambda^2\tau)$ in (4.8) as before, using Taylor's formula, we obtain the required asymptotic expansion

$$I = \sum_{m=0}^{\infty} \frac{\mu_m(t)}{m!}\, {}_qC_{j,\,j}^{2m+i,\,n}, \quad (t \to 0) \tag{4.10}$$

$${}_0C_{j,\,k}^{i,\,n} = A_{j,\,k}^{i,\,n}, \quad {}_1C_{j,\,k}^{i,\,n} = B_{j,\,k}^{i,\,n} = \int_0^\infty \lambda^i J_k(\lambda r)\, \omega_{jn} e^{-\lambda z}\, d\lambda \tag{4.11}$$

Mathematical induction is used to show that

$$\mu_m(t) = (-1)^m \sum_{k=0}^{m} \sum_{l=0}^{m-k} \frac{(-1)^{k-l}\, m!}{(m-k-l)!}\, t^{m-k-l} f_{k-l-2}(z, t) \tag{4.12}$$

where $f_k(z, t)$ are given by (4.4).

Finally we obtain the approximate formulas (asymptotically exact as $t \to 0$) for computing the stresses

$$\frac{\sigma_{rr}}{D} = -T + ht \sum_{n=0}^{N} (F_1{}^n - 2B_{1,\,n}^{2,\,n} - zB_{1,\,n}^{3,\,n}) + \tag{4.13}$$
$$\sum \Big\{ \Big[\frac{1}{r} A_{1,\,n+1}^{2m-2,\,n} + \frac{n^2 - n}{r^2} A_{1,\,n}^{2m-1,\,n} - A_{1,\,n}^{2m-3,\,n} \Big] f_{k-l-2}(z, t) +$$
$$[F_2{}^n - 2hB_{1,\,n}^{2m-2,\,n} - (1 - hz) B_{1,\,n}^{2m-3,\,n} + zB_{1,\,n}^{2m-4,\,n}] f_{k-l-2}(0, t) \Big\}$$

$$\frac{\sigma_{\varphi\varphi}}{D} = -T - ht \sum_{n=0}^{N} (F_1{}^n - 2\nu B_{1,\,n}^{2,\,n}) - \sum \Big\{ \Big[\frac{1}{r} A_{1,\,n+1}^{2m-2,\,n} + \frac{n^2 - n}{r^2} A_{1,\,n}^{2m-1,\,n} \Big] f_{k-l-2}(z, t) -$$
$$[F_2{}^n + 2\nu (hB_{1,\,n}^{2m-2,\,n} + B_{1,\,n}^{2m+3,\,n})] f_{k-l-2}(0, t) \Big\}$$

$$\frac{\sigma_{zz}}{D} = htz \sum_{n=0}^{N} B_{1,\,n}^{3,\,n} + \sum \{ A_{1,\,n}^{2m-3,\,n} f_{k-l-2}(z, t) -$$
$$[(1 + hz) B_{1,\,n}^{2m-3,\,n} + zB_{1,\,n}^{2m-4,\,n}] f_{k-l-2}(0, t) \}$$

$$\frac{\sigma_{r\varphi}}{D} = ht \sum_{n=0}^{N} \frac{n}{r^2} [2(n-1)(\nu - 1) B_{2,\,n}^{0,\,n} + (n-1) zB_{2,\,n}^{1,\,n} -$$
$$2r(\nu - 1) B_{2,\,n-1}^{1,\,n} - rzB_{2,\,n-1}^{2,\,n}] + \sum \frac{n}{r^2} \{ [(n-1) A_{2,\,n}^{2m-1,\,n} -$$
$$rA_{2,\,n-1}^{2m-2,\,n}] f_{k-l+2}(z, t) - [2h(n-1)(\nu - 1) B_{2,\,n}^{2m,\,n} -$$
$$(n-1)(1 - 2\nu - hz) B_{2,\,n}^{2m-1,\,n} + (n-1) zB_{2,\,n}^{2m-2,\,n} -$$
$$2h(\nu - 1) rB_{2,\,n-1}^{2m-1,\,n} + (1 - 2\nu - hz) rB_{2,\,n+1}^{2m+2,\,n} - rzB_{2,\,n-1}^{2m+3,\,n}] f_{k-l+2}(0, t) \}$$

$$\frac{\sigma_{rz}}{D} = ht \sum_{n=0}^{N} \Big[\frac{n}{r} (B_{2,\,n}^{1,\,n} - zB_{1,\,n}^{2,\,n}) - B_{1,\,n-1}^{2,\,n} + zB_{1,\,n+1}^{3,\,n} \Big] +$$
$$\sum \Big\{ \Big(\frac{n}{r} A_{1,\,n}^{2m+1,\,n} - hA_{1,\,n-1}^{2m-2,\,n} \Big) [f_{k-l-2}(z, t) - \psi_{k-l+2}(z, t)] -$$
$$\Big[\frac{n}{r} (hB_{1,\,n}^{2m-1,\,n} - hzB_{1,\,n}^{2m+2,\,n} - zB_{1,\,n}^{2m-3,\,n}) - hB_{1,\,n-1}^{2m-2,\,n} + hzB_{1,\,n+1}^{2m-3,\,n} + zB_{1,\,n-1}^{2m-4,\,n} \Big] f_{k-l-2}(0, t) \Big\}$$

$$\frac{\sigma_{\varphi z}}{D} = ht\sum_{n=0}^{N}\frac{n}{r}\left(B_{2,n}^{1,n} - zB_{2,n}^{2,n}\right) + \sum\frac{n}{r}\{A_{2,n}^{2m+1,n}[f_{k+l+2}(z,t) - \psi_{k+l+2}(z,t)] - (hB_{2,n}^{2m+1,n} - hzB_{2,n}^{2m+2,n} - zB_{2,n}^{2m+3,n})f_{k-l+3}(0,t)\}$$

where

$$F_1{}^n = \frac{2(\nu-1)}{r}B_{1,n+1}^{1,n} + \frac{z}{r}B_{1,n+1}^{2,n} + \frac{n^2-n}{r^2}[2(\nu-1)B_{1,n}^{0,n} + zB_{1,n}^{1,n}] \tag{4.14}$$

$$F_2{}^n = -\frac{2h(\nu-1)}{r}B_{1,n+1}^{2m+1,n} + \frac{1-2\nu-hz}{r}B_{1,n+1}^{2m+2,n} - \frac{z}{r}B_{1,n+1}^{2m+3,n} - \frac{n^2-n}{r^2}[2h(\nu-1)B_{1,n}^{2m,n} + zB_{1,n}^{2m+2,n} - (1-2\nu-hz)B_{1,n}^{2m+1,n}]$$

$$\psi_k(z,t) = -\frac{\partial\varphi_{2k+1}}{\partial t}, \quad \sum = \sum_{n=0}^{N}\sum_{m=0}^{M}\sum_{k=0}^{m}\sum_{l=0}^{m-k}\frac{(-1)^{m+k+l}t^{m-k-l}}{(m-k-l)!}$$

(the functions $\varphi_k(z,t)$ are defined in (4.5)). Note that the integrals (4.11) converge at least as rapidly as the integrals (4.4).

5. Let us assess the error of the approximate solution (4.7), (4.13). To obtain a uniform estimate, we make use of the fact that $\partial f_k/\partial t = f_{k-1}$, and $f_1(z,t)$ is a solution of the one-dimensional problem of heat conduction for a half-space with zero initial temperature, when convective heat exchange occurs when $t > 0$ at the boundary of the half-space separating it from a medium at unit temperature, i.e. $0 \leqslant f_1(z,t) < 1$ is a function that increases monotonically with t. Integrating by parts and applying the mean value theorem, we obtain

$$0 \leqslant \int_0^t \tau^m f_0(z,\tau)\,d\tau < t^{m-1}f_1(z,t), \tag{5.1}$$

$$0 \leqslant \int_0^t d\vartheta\int_0^{\vartheta}\tau^m f_0(z,\tau)\,d\tau < t^{m-1}f_1(z,t)$$

Using (5.1), the well-known inequalities $|J_n(x)| \leqslant 1/\sqrt{2}$, $n = 1, 2, \ldots$; $|J_0(x)| \leqslant 1$ /11/ and the inequality $0 \leqslant \lambda x e^{-\lambda x} \leqslant e^{-1}$, we can obtain the required estimate. Omitting the cumbersome calculations, we give the final result for the estimates of the errors in the M-th approximation to the temperature and stresses, corresponding to deletion from the asymptotic expansions of all terms from the $M-1$-th term on. We have

$$|\delta_T^M| \leqslant \frac{t^{M-1}f_1(z,t)}{(M-1)!}\sum_{n=0}^{N}A_nI_n^{2M-3} \tag{5.2}$$

$$|\delta_{rr}^M| \leqslant \beta_2\gamma\sum_{n=0}^{N}(A_n + B_n)\alpha_{1n}, \quad |\delta_{\varphi\varphi}^M| \leqslant \beta_2\sum_{n=0}^{N}(\gamma B_n + A_n)\alpha_{1n}$$

$$|\delta_{zz}^M| \leqslant \beta_2\sum_{n=0}^{N}A_n(he^{-1}I_n^{2M-4} + \gamma I_n^{2M-5}), \quad |\delta_{r\varphi}^M| \leqslant \beta_2\gamma\sum_{n=0}^{N}E_n\alpha_{1n}$$

$$|\delta_{rz}^M| \leqslant \beta_1\sum_{n=0}^{N}C_n\alpha_{2n}, \quad |\delta_{\varphi z}^M| \leqslant \beta_1\sum_{n=0}^{N}D_n\alpha_{2n}$$

$$\gamma = 2 - e^{-1}, \quad \beta_1 = Dt^{M-2}/(M-1)!, \quad \beta_2 = \beta_1 f_1(0,t),$$

$$\alpha_{1n} = I_n^{2M-5} - hI_n^{2M-4}$$

$$\alpha_{2n} = (h\gamma I_n^{2M-4} - e^{-1}I_n^{2M-5})f_1(0,t) + hI_n^{2M-4}$$

$$I_n{}^i = \int_0^\infty \lambda^i([\vartheta_n{}^H(\lambda)]^2 + [\tau_n{}^H(\lambda)]^2)^{1/2}\,d\lambda, \quad i = 3, 4.$$

where

$$A_1 = A_2 = A_n = 2B_1 = B_n = C_0 = C_n = D_2 = C_2 = D_n = 2E_1 = 2E_n = 1/\sqrt{2} \quad (n = 3, 4, 5, \ldots)$$

$$A_0 = 1, \quad B_0 = C_1 = D_1 = 2E_1 = 2E_2 = (1 + \sqrt{2})/2\sqrt{2}$$

$$B_2 = (3 - \sqrt{2})/4\sqrt{2}, \quad D_0 = E_0 = 0$$

6. Let us study in more detail the behaviour of the temperature and the stresses for short times.

The zero approximation to the temperature

$$T = \Theta(r,\varphi)f_1(z,t) \tag{6.1}$$

is asymptotically exact as $t \to 0$, in the sense that the relative error of the approximate solution (6.1) tends to zero as $t \to 0$.

From the physical point of view this means that at short heating times the limiting distribution of the temperature propagates in the direction of the Oz axis without "diffusing" in the radial direction and retaining its "form".

In particular, if the limiting temperature distribution has a first-order discontinuity (a jump) along some line, then the temperature field in the half-space determined by the solution of the heat conduction equation, is continuous, but the jump is smoothed out only by the infinitesimals in t.

In the case of the stresses we can find coarser approximate formulas which will still remain asymptotically exact in the same sense. Such are e.g. the relations obtained from (4.13) by deleting the terms containing the symbol Σ, or, which amounts to the same, by deleting from the integrands in (3.6) terms containing ω. The errors in the approximate formulas obtained in this manner are estimated by formal substitution of the value $M = -1$ into (5.2).

Let us also consider the often-used approximation

$$\sigma_{rr} = \sigma_{\varphi\varphi} = -\frac{2\alpha\mu(1+\nu)}{1-\nu} T \tag{6.2}$$

which is obtained by retaining, on the right-hand sides of formulas (4.13) for σ_{rr} and $\sigma_{\varphi\varphi}$ the term $-T$ only. Formulas (5.2) imply directly that the error of the approximation (6.2) is of the order of t as $t \to 0$. A more accurate estimate obtained using the scheme described above shows that the approximation (6.2) is asymptotically exact only on the surface $z = 0$ of the half-space, and at the remaining points the relative error of relations (6.2) tends to unity (to 100%) as $t \to 0$. It is precisely this that leads to the error in the sign of the stresses (i.e. not only to a quantitative, but also to a qualitative error) as shown in /5/.

7. We consider, as an example, the boundary temperature distribution in dimensional coordinates

$$\Theta(r, \varphi) = \Theta_0 \left[\frac{a^3}{(r^2+\delta^2)^{3/2}} + \frac{b^4 r}{(r^2+\delta^2)^{5/2}} \cos\varphi \right] \tag{7.1}$$

where a, b, δ are constants with dimension of length, and Θ_0 is the temperature at the region of coordinates.

The distribution (7.1) has a bell-like shape and can be used to model the real temperature distribution at the boundary of a body being fractured by a high temperature gas jet /4/ when the jet is directed at an angle to the surface. If this angle is equal to $\pi/2$, i.e. if we have axial symmetry, then $b = 0$ in (7.1), otherwise the quantity b^4/a^3 characterizes the asymmetry of the distribution.

Passing further to dimensionless coordinates we put $\delta = 1$ in (7.1), in accordance with Sect.2 and regard a^3 as a^3/δ^3 and b^4 as b^4/δ^4. Two terms remain in the Fourier series ($N = 1$), and using the formulas from /12/ we obtain

$$\vartheta_0^H(\lambda) = \Theta_0 a^3 e^{-\lambda}, \quad \vartheta_1^H(\lambda) = {}^1/_3\Theta_0 b^4 \lambda e^{-\lambda} \tag{7.2}$$

Now we can find the coefficients of the asymptotic expansion (4.13) using the integral from /12/, which can conveniently be reduced to the form

$$\int_0^\infty \lambda^i e^{-\lambda x} J_j(\lambda r)\, d\lambda = (-1)^i \frac{1}{r^j} \frac{d^i}{dx^i} \left[\frac{(\sqrt{x^2+r^2}-x)^j}{\sqrt{x^2+r^2}} \right], \quad i = 0, 1, 2, \ldots \tag{7.3}$$

where the function itself represents its zero order derivative by definition. The coefficients A_{ij} and B_{ij} can be found by putting $x = 1$ and $x = 1 + z$ respectively in (7.3)

$$\int_0^\infty \lambda^k e^{-\lambda}\, d\lambda = k!, \quad k = 0, 1, 2, \ldots \tag{7.4}$$

Thus in the case of distribution (7.1) the final expressions contain only a single special function $\operatorname{erfc} z$, which has simple approximating and asymptotic formulas over the whole interval of variation of the argument. The computations carried out with the help of (4.4), (4.13), (5.2), (7.3) and (7.4) are elementary.

8. In the axisymmetric case the relations obtained above agree with the results of /5/. Note that there are misprints in formulas (3.4) of /5/: $\sigma_{zz} = thB_{s0} + \ldots$ should read $\sigma_{zz} = htzB_{s0} + \ldots$ and the factor t^{n-k} is missing from the relations for $\varphi_n(z, t)$ under the summation sign.

REFERENCES

1. KILL I.D., On thermoelastic stresses within a half-space. Inzh. zh. MTT, 1, 1966.
2. DUDOLADOV L.S., On thermal fracturing of rocks. Fiz.-tekhn. probl. razrabotki poleznykh iskopaemykh, 2, 1969.
3. GERMANOVICH L.N. and KILL I.D., On thermal stresses in an elastic half-space. PMTF, 3, 1983.
4. DMITRIYEV A.P. and GONCHAROV S.A., Thermodynamic Processes in Rocks. Moscow, Nedra, 1983.
5. CHEREPANOV G.P., Mechanics of brittle fracture. N.Y. McGraw-Hill, 1979.

6. NIKIFOROVSKII V.S. and SERYAKOV V.M., On the problem of the thermal and stress state and fracturing of composite bodies on heating. Izv. SO Akad. Nauk SSSR, Ser. tekhn. Nauk, 1, 3, 1975.
7. PARKUS G., Unsteady thermal stresses. Moscow, Fizmatgiz, 1963.
8. NOWACKI W., Thermoelasticity. Reading, Mass, Addison-Wesley, 1962.
9. EVGRAFOV M.A., Asymptotic Estimates and Entire Functions. Moscow, Nauka, 1979.
10. LAVRENT'EV M.A. and SHABAT B.V., Methods of the Theory of Functions of a Complex Variable. Moscow, Nauka, 1965.
11. Manual of Special Functions with Formulas, Graphs and Mathematical Tables. Moscow, Nauka, 1979.
12. GRADSHTEIN I.S. and RYZHIK I.M., Tables of Integrals, Sums, Series and Products. Moscow, Nauka, 1971.

Translated by L.K.

PMM U.S.S.R., Vol.49, No.6, pp.770-777, 1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

ON USING THE MORE-ACCURATE EQUATIONS OF THIN COATINGS IN THE THEORY OF AXISYMMETRIC CONTACT PROBLEMS FOR COMPOSITE FOUNDATIONS*

V.I. AVILKIN, V.M. ALEKSANDROV and E.V. KOVALENKO

More-accurate equations describing the axisymmetric deformations of elastic, thin-walled elements (coatings) are derived using the asymptotic analysis of the solution to the first fundamental problem of the theory of elasticity for a layer. The notable difference distinguishing these relations from the classical, Kirchhoff-Love and Reissner-Timoshenko equations of flexure of plates, and their modifications /1/, is, that there are no concentrated forces at the edges of the stamp when the corresponding contact problems are solved. Moreover, the formulas obtained contain the equations of classical theory as a special case. The solutions obtained using various applied theories are compared with the corresponding solution obtained using the equations of the theory of elasticity, using the example of the axisymmetric contact problem of impressing a plane circular stamp into a layer lying on a Fuss-Winkler foundation. The characteristic parameters of the problem in question are computed by numerical methods.

1. As we know /2/, the solution of the equations of the theory of elasticity can be expressed, in the case of axisymmetric problems, by a single biharmonic function $\chi(r, z)$

$$\Delta^2\chi = 0 \quad \left(\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) \tag{1.1}$$

$$2Gu = -\frac{\partial^2\chi}{\partial r\partial z}, \quad 2Gw = \left[2(1-\nu)\Delta - \frac{\partial^2}{\partial z^2}\right]\chi + D \tag{1.2}$$

$$\sigma_z = \frac{\partial}{\partial z}\left[2(1-\nu)\Delta - \frac{\partial^2}{\partial z^2}\right]\chi, \quad \tau_{rz} = \frac{\partial}{\partial z}\left[(1-\nu)\Delta - \frac{\partial^2}{\partial z^2}\right]\chi \tag{1.3}$$

Let us consider the first boundary value problem on the equilibrium of an elastic layer of thickness $2h$, when the application of external loads deforms it symmetrically about the z-axis. We shall seek the solution of (1.1) in the form of a Hankel integral /3/

$$\chi = \int_0^\infty \xi\Phi(\xi, z) J_0(r\xi)\,d\xi, \quad \Phi = \int_0^\infty r\chi(r, z) J_0(r\xi)\,dr \tag{1.4}$$

Substituting (1.4) into (1.1) and carrying out simple mathematical reduction, we obtain

$$\Phi(\xi, z) = (c_1 \operatorname{ch}\xi z + c_2\xi z \operatorname{sh}\xi z + d_1 \operatorname{sh}\xi z + d_2\xi z \operatorname{ch}\xi z)\,\xi^{-3} \tag{1.5}$$

where c_j and d_j $(j = 1, 2)$ are functions of ξ whose form is determined from the boundary conditions of the problem in question

$$\sigma_z(r, h) = \sigma_+(r), \quad \tau_{rz}(r, h) = \tau_+(r) \tag{1.6}$$
$$\sigma_z(r, -h) = \sigma_-(r), \quad \tau_{rz}(r, -h) = \tau_-(r)$$
$$\sigma_z, \ \tau_{rz} \to 0, \quad (r^2 + z^2) \to \infty$$

*Prikl.Matem.Mekhan., 49, 6, 1010-1018, 1985